

JLR-06/10

MATHEMATICS

Time allowed : Three Hours

Full Marks : 100

*All questions carry equal marks.**Answer any FIVE questions.**(Symbols used have their usual meaning)*

1. (a) If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that :

(i) $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$ and

- (ii) $\lim_{n \rightarrow \infty} a_n = 0$ show that the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \text{ is convergent.} \quad 8$$

- (b) (i) Define (C, 2) summability. Show that :

$1, -1, 2, -2, 3, -3, \dots$ is not (C, 1) summable but (C, 2) summable. 6

- (ii) Determine all real values of x for which the following series converges :

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) \frac{\sin nx}{n}. \quad 6$$

2. (a) Let $f(z)$ be an analytic function in a simply connected domain D enclosed by a rectifiable Jordan curve

C and let $f(x)$ be continuous on C . Show that

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

where z_0 is any point in D . 8

- (b) (i) Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$. 6

- (ii) Represent the function $f(z) = \frac{z}{(z-1)(z-3)}$ by a series of negative powers of $(z-1)$ which converges to $f(z)$ when $0 < |z-1| < 2$. 6

3. (a) State and prove Lebesgue's monotone convergence theorem. 8

- (b) (i) Construct a non-measurable set on the real line. 6

- (ii) Prove that the function $\frac{\sin x}{x}$ is not Lebesgue integrable over $[0, \infty)$. 6

4. (a) State and prove Gauss-Bonnet Theorem on the total curvature. 8

- (b) (i) On the paraboloid $x^2 - y^2 = z$, find the orthogonal trajectories of the sections by the planes $z = \text{constant}$. 6

- (ii) Obtain the curvature and torsion of the curve of intersection of two quadratic surfaces :

$$ax^2 + by^2 + cz^2 = 1, a'x^2 + b'y^2 + c'z^2 = 1$$

6

5. (a) Using Gaussian elimination find the inverse of the matrix :

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$

8

- (b) (i) Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during the 5 years.

6

Year	2005	2006	2007	2008	2009
Export (in tons)	443	384	—	397	467

- (ii) Using the fourth order Runge-Kutta method, find an approximate value of y when $x = 0.2$ given that $y' = x + y$, $y(0) = 1$.

6

6. (a) Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \rightarrow 0$. Show that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

8

- (b) (i) Let X be a second countable space. If a non-empty open set G in X is represented as the union of a class $\{G_i\}$ of open sets, show that G can be represented as a countable union of G_i 's. 6
- (ii) Prove that any continuous image of a connected space is connected. 6
7. (a) State and prove the Closed Graph Theorem. 8
- (b) (i) Prove that a normed space X is finite dimensional if and only if the closed unit sphere in X is compact. 6
- (ii) Consider the Banach space $(C[0, 1], \|\cdot\|_\infty)$ and the normed space $(C^1[0, 1], \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the sub norm. Define the mapping $T : C^1[0, 1] \rightarrow C[0, 1]$ by $(Tx)(t) = x'(t)$, $x \in C^1[0, 1]$. Prove that (1) T is not bounded (2) T is closed. 6
8. (a) Let $J(i)$ denote the set of all complex numbers of the form $a + ib$ where a, b are integers. Prove that $J[i]$ is an Euclidean ring. 8
- (b) (i) Prove that the ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R if and only if a_0 is a prime element of R . 6
- (ii) If L is an algebraic extension of K and if K is an algebraic extension of F , show that L is an algebraic extension of F . 6

9. (a) Let V be a finite dimensional vector space over the field F , and let W be a subspace of V . Prove that $\dim W + \dim W^\circ = \dim V$. 8
- (b) (i) Let A be any $m \times n$ matrix over the field F . Show that the row rank of A is equal to the column rank of A . 6
- (ii) If F is a field, and M is any non-zero ideal in $F[x]$, show that there is a unique monic polynomial d in $F[x]$ such that M is the principal ideal generated by d . 6
10. (a) Using Monge's method, solve the following equation : 8
- $$2x^2r - 5xys + 2y^2t + 2(px + qy) = 0.$$
- (b) (i) Establish the orthogonal property of Legendre's polynomials. 6
- (ii) Find the inverse Laplace transform of : 6
- $$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}.$$